Special Situations in the Simplex Algorithm

Degeneracy

Consider the linear program:

$$
\begin{array}{llrlrl}
\text{Maximize} & 2x_1 & +x_2 & & & \\
\text{Subject to:} & & & & & \\
 & 4x_1 & +3x_2 & \leq & 12 & (1) \\
 & 4x_1 & +x_2 & \leq & 8 & (2) \\
 & 4x_1 & +2x_2 & \leq & 8 & (3) \\
 & x_1,\, x_2 \geq 0\,. & & & &
\end{array}
$$

We will first apply the Simplex algorithm to this problem. After a couple of iterations, we will hit a degenerate solution, which is why this example is chosen. We will then examine the geometrical origin of degeneracy and the related issue of "cycling" in the Simplex algorithm, with the help of the graphical representation of this problem.

After introducing three slack variables and setting up the objective function, we obtain the following initial Simplex tableau.

Basic Variable	z	x_1	x_2	s_1	s_2	s_3	
	1	−2	−1	0	0	0	0
s_1	0	4	3	1	0	0	12
s_2	0	4	1	0	1	0	8
s_3	0	4	2	0	0	1	8

With x_1 as the entering variable, there is a tie for the minimum ratio, at R_2 and R_3. This (also observed in the previous two-phase example) implies that after a pivot with either R_2 or R_3 as the pivot row, the resulting tableau will have a degenerate basic variable. Let us choose R_2 (say) as the pivot row. Then, after executing a pivot, we obtain the tableau below.

Tableau I:

Basic Variable	z	x_1	x_2	s_1	s_2	s_3	
	1	0	−1/2	0	1/2	0	4
s_1	0	0	2	1	−1	0	4
x_1	0	1	1/4	0	1/4	0	2
s_3	0	0	1	0	−1	1	0

The current basic feasible solution is $(x_1, x_2, s_1, s_2, s_3) = (2, 0, 4, 0, 0)$, where s_3 is (as expected) a degenerate basic variable. The next pivot column and pivot row will be the x_2-column and R_3, respectively. After executing another pivot, we obtain the following tableau.

Tableau II:

Basic Variable	z	x_1	x_2	s_1	s_2	s_3	
	1	0	0	0	0	1/2	4
s_1	0	0	0	1	1	-2	4
x_1	0	1	0	0	1/2	$-1/4$	2
x_2	0	0	1	0	-1	1	0

Again, the current basic feasible solution is $(x_1, x_2, s_1, s_2, s_3) = (2, 0, 4, 0, 0)$. However, the identify of the degenerate basic variable has switched from s_3 to x_2. Note that this tableau happens to be optimal (independent of the phenomenon of degeneracy).

To understand what it means to have a degenerate solution, let us now refer to the graphical representation of this problem, which is shown in Figure LP-8. Notice that three, not two, constraint equations pass through the corner-point solution $(x_1, x_2) = (2, 0)$. These equations are: $x_2 = 0$, $4x_1 + x_2 = 8$, and $4x_1 + 2x_2 = 8$. Since only two lines are needed to define such an intersection, we see that degeneracy is a manifestation of redundancy in information. That is, we can choose to let any pair of these equations (out of

$$\binom{3}{2} = 3$$

combinations) to define this intersection. For example, if we choose $x_2 = 0$ and $4x_1 + x_2 = 8$ as the defining equations, then, since the solution to this pair of equations will automatically satisfy equation $4x_1 + 2x_2 = 8$, the value of the slack variable associated with the inequality $4x_1 + 2x_2 \leq 8$, namely s_3, must turn out to be 0. This accounts for the appearance of the degenerate basic variable s_3 in Tableau I. Similarly, if we choose $4x_1 + x_2 = 8$ and $4x_1 + 2x_2 = 8$ as the defining equations, then the inequality constraint $x_2 \geq 0$ will turn out to be binding. This accounts for the fact that x_2 is a degenerate basic variable in Tableau II.

What will happen if we choose $x_2 = 0$ and $4x_1 + 2x_2 = 8$ as the defining equations? A careful examination of Tableau II shows that if we choose the s_2-column and R_3 as the pivot column and the pivot row, then the following tableau results after a pivot.

Tableau III:

Basic Variable	z	x_1	x_2	s_1	s_2	s_3	
	1	0	0	0	0	1/2	4
s_1	0	0	1	1	0	-1	4
x_1	0	1	1/2	0	0	1/4	2
s_2	0	0	-1	0	1	-1	0

This new tableau, again, corresponds to the solution $(x_1, x_2, s_1, s_2, s_3) = (2, 0, 4, 0, 0)$. Notice however that the slack variable s_2 associated with the inequality $4x_1 + x_2 \leq 8$ has indeed replaced x_2 as the degenerate basic variable.

Recall that in the last pivot, the pivot element is a negative number, -1. This will never occur during ordinary Simplex iterations. (Why?) Our purpose for carrying out such a pivot is to show that there is indeed a third tableau that is associated with the corner-point solution (2, 0).

Now, with three tableaus all corresponding to the same set of coordinates, the question is: Is it possible for the Simplex algorithm to cycle through these (or a subset of these) tableaus forever? Theoretically, the answer is yes. However, this happens rarely in practice, and can in fact be avoided. Observe that in order for the Simplex algorithm to cycle, we must repeat ourselves during the iterations. Suppose a (nonoptimal) tableau that has been visited before is generated as the result of a pivot. The idea is to look for a different choice for either the pivot column or the pivot row. Whenever such a choice can be found, we simply continue the algorithm with the new choice. It turns out that repeatedly doing this will eventually take us out of any degenerate solution. We will leave out the difficult theoretical argument that supports this last assertion.

In terms of the mechanics of the Simplex algorithm, how does one get out of degeneracy? We will answer this with an example. Consider a tableau that has the configuration below.

	Pivot Column		RHS	
	a		d	
	?		?	
	?		?	
$\cdots$	b	$\cdots$	0	← Degeneracy
	c		e	← Pivot Row
	?		?	
	?		?	

Here, we assume that: we are maximizing, the column explicitly shown on the left is the pivot column, the entry a is negative, the entry b is nonpositive, the entry c is positive, the column shown on the right is the right-hand-side column, the entry e is positive, and finally the row containing c and e is the pivot row. Notice that in the row just above the pivot row, the right-hand-side constant equals 0, which indicates that the current solution is degenerate. However, since b is assumed to be nonpositive, we do not compute a ratio for this row in the ratio test. (To simplify discussion, we also assume that this is the only row with a zero on the right-hand side.) This makes it possible for a row with a positive right-hand-side constant to become the pivot row. Now, when a pivot is performed with the entry c as the pivot element, we will multiply the pivot row by $-a/c$ and add the outcome into R_0. This will result in a strict improvement in the objective-function value, from d to $d + (-a/c)e$. With this strict increase, we are now guaranteed never to return to this tableau again in the remainder of the Simplex algorithm.

In summary, the phenomenon of cycling in the Simplex algorithm is caused by degeneracy. While cycling can be avoided, the presence of degenerate solutions may temporarily suspend progress in the algorithm.

Unboundedness

Consider the linear program:

$$
\begin{array}{lrrcrr}
\text{Maximize} & 2x_1 & +x_2 & & & \\
\text{Subject to:} & & & & & \\
 & x_1 & -x_2 & \leq & 10 & (1) \\
 & 2x_1 & -x_2 & \leq & 40 & (2) \\
 & \multicolumn{5}{l}{x_1,\, x_2 \geq 0\,.}
\end{array}
$$

Again, we will first apply the Simplex algorithm to this problem. The algorithm will take us to a tableau that indicates unboundedness of the problem. We will then examine the geometrical origin of unboundedness with the help of the graphical representation of this problem.

After introducing two slack variables and setting up the objective function, we obtain the following initial Simplex tableau.

Basic Variable	z	x_1	x_2	s_1	s_2	
	1	−2	−1	0	0	0
s_1	0	1	−1	1	0	10
s_2	0	2	−1	0	1	40

With x_1 as the entering variable, it is easily seen that R_1 is the pivot row. After executing a pivot, we obtain the tableau below.

Basic Variable	z	x_1	x_2	s_1	s_2	
	1	0	−3	2	0	20
x_1	0	1	−1	1	0	10
s_2	0	0	1	−2	1	20

Since x_2 has a negative coefficient in R_0, this tableau is not optimal. Another pivot takes us to the next tableau.

Basic Variable	z	x_1	x_2	s_1	s_2	
	1	0	0	−4	3	80
x_1	0	1	0	−1	1	30
x_2	0	0	1	−2	1	20

This tableau again is not optimal. However, at this point, we are unable to perform further iterations, because as we attempt to carry out a ratio test with s_1 as the entering variable, it turns out that there is no ratio to compute. What this means is that as we attempt to bring s_1 in as a basic variable, none of the constraints will stop us from increasing its value to infinity. Now, as the value of s_1 increases, the objective-function value will also increase correspondingly at a rate of 4. It follows that the problem does not have an optimal solution.

The feasible region of this problem is depicted in Figure LP-9. There, we see that the Simplex algorithm starts with the point (0, 0), follows the x_1-axis to the point (10, 0), rises diagonally to the point (30, 20), and then takes off to infinity along an infinite "ray" that emanates from (30, 20).

More formally, what we have is that for any nonnegative δ, the solution $(x_1, x_2, s_1, s_2) = (30+\delta, 20+2\delta, \delta, 0)$ is feasible. Since this solution has a corresponding objective-function value of $80+4\delta$, we see that the problem is unbounded.

Clearly, unboundedness of a problem can occur only when the feasible region is unbounded, which, unfortunately, is something we cannot tell in advance of the solution attempt. In the above example, we detected unboundedness when we encountered a pivot column that does not contain any positive entry. More generally, we can in fact declare a problem as unbounded if *any* (nonbasic) column, not necessarily associated with the entering variable, is identified to have the above-stated property at the end of an iteration. Referring back to the initial tableau, we see that, indeed, the x_2-column had this property. Therefore, we could have concluded that the problem is unbounded at the outset. The difference is that the algorithm would then follow the x_2-axis to infinity. (Of course, another difference is the amount of effort.)

The corresponding condition for unboundedness in a minimization problem is slightly different: We should look for a nonbasic column with a positive coefficient in R_0 and with all other entries nonpositive.

In most applications of linear programming, if a problem turns out to be unbounded, it is often due to the fact that at least one relevant constraint has been left out during the formulation stage. Therefore, one should carefully reexamine the original formulation.

Multiple Optimal Solutions

Consider the linear program:

$$\begin{array}{llllll} \text{Maximize} & 4x_1 & +14x_2 & & & \\ \text{Subject to:} & & & & & \\ & 2x_1 & +7x_2 & \leq & 21 & (1) \\ & 7x_1 & +2x_2 & \leq & 21 & (2) \\ & x_1,\ x_2 \geq 0\,. & & & & \end{array}$$

As before, we will first apply the Simplex algorithm to this problem. The algorithm will take us to a tableau that indicates that alternative optimal solutions exist. We will then examine the geometrical origin behind the existence of alternative optimal solutions, with the help of the graphical representation of this problem.

After introducing two slack variables and setting up the objective function, we obtain the following initial Simplex tableau.

Basic Variable	z	x_1	x_2	s_1	s_2	
	1	-4	-14	0	0	0
s_1	0	2	7	1	0	21
s_2	0	7	2	0	1	21

With x_2 as the entering variable, it is easily seen that R_1 is the pivot row. After executing a pivot, we obtain the tableau below.

Basic Variable	z	x_1	x_2	s_1	s_2	
	1	0	0	2	0	42
x_2	0	$2/7$	1	$1/7$	0	3
s_2	0	$45/7$	0	$-2/7$	1	15

At this point, since every nonbasic variable has a nonnegative coefficient in R_0, the current solution $(x_1, x_2, s_1, s_2) = (0, 3, 0, 15)$ is optimal. However, notice that the nonbasic variable x_1 has a coefficient of 0 in R_0. This implies that if we attempt to let x_1 enter the basis, then the objective-function value will not change. Indeed, after a pivot with the x_1-column as the pivot column, we obtain the following new tableau.

Basic Variable	z	x_1	x_2	s_1	s_2	
	1	0	0	2	0	42
x_2	0	0	1	$7/45$	$-2/45$	$7/3$
x_1	0	1	0	$-2/45$	$7/45$	$7/3$

With the same objective-function value, the new solution $(x_1, x_2, s_1, s_2) = (7/3, 7/3, 0, 0)$ is, of course, also optimal. Note that a further attempt at a pivot in the s_2-column will take us back to the previous solution. We will therefore not pursue things further.

The feasible region of this problem is depicted in Figure LP-10. There, we see that the Simplex algorithm starts with the point (0, 0), travels along the x_2-axis to the first optimal solution at (0, 3), and then continues on to the second optimal solution at (7/3, 7/3). Notice that the objective-function line $4x_1 + 14x_2 = c$ (for any c) is parallel to the edge that begins at (0, 3) and ends at (7/3, 7/3). Hence, every point on this edge is optimal.

In general, if we are given two optimal solutions to a linear program, then an infinite number of optimal solutions can be constructed. In this example, both (0, 3) and (7/3, 7/3) are optimal. Therefore, every point on the edge connecting these two points will also be optimal. Formally, points on this edge are traced out by solutions of the form:

$$(x_1, x_2) = \delta \times (0, 3) + (1 - \delta) \times (7/3, 7/3),$$

where δ is any value in the interval $[0, 1]$. As specific examples, if we let $\delta = 1$, then we have the point $(0, 3)$; if we let $\delta = 0$, then we have the point $(7/3, 7/3)$; and if we let $\delta = 1/2$, then we have the point $(7/6, 8/3)$, which is half way between $(0, 3)$ and $(7/3, 7/3)$.